



TITLE:

ON 1-BRIDGE TORUS KNOTS (On Heegaard Splittings and Dehn surgeries of 3-manifolds, and topics related to them)

AUTHOR(S):

Choi, Doo Ho; Ko, Ki Hyoung

CITATION:

Choi, Doo Ho ...[et al]. ON 1-BRIDGE TORUS KNOTS (On Heegaard Splittings and Dehn surgeries of 3-manifolds, and topics related to them). 数理解析研究所講究録 2001, 1229: 19-32

ISSUE DATE:

2001-09

URL:

<http://hdl.handle.net/2433/41440>

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ON 1-BRIDGE TORUS KNOTS

DOO HO CHOI AND KI HYOUNG KO

ABSTRACT. A 1-bridge torus knot is a knot drawn on a standard torus in S^3 with 1-bridge. We introduce two types of normal forms to parametrize the family of 1-bridge torus knots that are similar to the Schubert's normal form and the Conway's normal form for 2-bridge knots. For a given Schubert's normal form we give a classificatoin of some sub-class of 1-bridge torus knots. We also give a description of the double brannced cover of S^3 branched along any 1-bridge torus knots by using the Conway's normal form and obtain an explicit formula for the first homology of the double cover.

1. INTRODUCTION

One of traditions in knot theory is to study a family of knots satisfying a certain condition. Examples of such families include the family of torus knots studied by Dehn and Schreier and the family of 2-bridge knots studies by Schubert, Montesinos and Conway. These classes can be referred as the classes of knots and links indexed by the pairs (g, b) of non-negative integers as defined in [9]. A knot K in a 3-manifold M has a (g, b) -decomposition or is called a (g, b) -knot if for some heegaard splitting $M = U \cup V$ of genus g , each of $K \cap U$ and $K \cap V$ is consisted of trivial b arcs. A collection of properly embedded arcs in a 3-manifold W with boundary is *trivial* if arcs α in the collection together with arcs on ∂W joining the two ends of the arcs bound mutually disjoint disks in W . A (g, b) -knot can be embedded in a heegaard surface of genus g in M except at b over(or under)-bridges and vice versa. Torus knots are $(1, 0)$ -knots and 2-bridge knots are $(0, 2)$ -knots. Clearly the family of (g, b) -knots becomes strictly larger as g or b increase. Since an over-bridge can

1991 *Mathematics Subject Classification.* 57M25, 57M27.

Key words and phrases. 1-bridge torus knot, Schubert's normal form, Schubert's normal form, double branched cover.

be removed by adding a handle and by embedding the over-bridge into the added handle, (g, b) -knots are contained in the family of $(g + 1, b - 1)$ -knots.

In this article we study the family of 1-bridge torus knots, that is, $(1, 1)$ -knots in S^3 . This family contains torus knots and 2-bridge knots and is contained in the family of double torus knots, that is, $(2, 0)$ -knots. Hill and Murasugi studied the family of double torus knots in [11, 12] and parametrized the family. Non-trivial knots with the trivial Alexander polynomial was found in the subfamily of double torus knots that separate the double torus. They also considered non-separating double torus knots and a subfamily of 1-bridge torus knots and found various double torus knots that are fibered.

The 1-bridge torus knot has the tunnel number one, but not all tunnel-number-one knots are 1-bridge torus knots. In [14], Morimoto, Sakuma and Yokota found tunnel-number-one knots that are not 1-bridge torus knots as confirmed by a condition on the Jones polynomial for a knot to admit a (g, b) -decomposition in [18]. In [15], they gave another criteria to determine whether a given knot has the tunnel number one and whether it is a 1-bridge torus knot.

Besides torus knots and 2-bridge knots, the family of 1-bridge torus knots includes Berge's double-primitive knots, 1-bridge braids that were classified by Gabai in [10] and satellite 1-bridge torus knots. Morimoto and Sakuma studied satellite 1-bridge torus knots and classified their unknotting tunnels in [13].

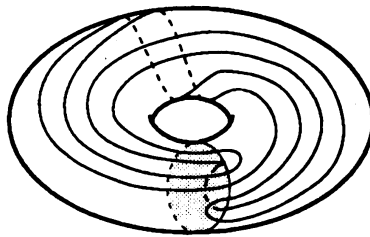


FIGURE 1. 1-bridge torus knot

In this article, we parameterize the family of 1-bridge torus knots using two kinds of normal forms as done for the family of 2-bridge knots. Schubert described a 2-bridge knots

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by a pair of integers of a certain condition from its top view. In the top view a 2-bridge knot is embedded in a plane except the two bridges. He in fact completely classified 2-bridge knots using this normal form [17]. Since a 1-bridge knot can be embedded in a standard torus except the bridge (See Figure 1), we will describe it by a 4-tuple of integers from this top view. We will call such a 4-tuple the *Schubert's normal form* of the 1-bridge torus knot determined by the 4-tuple. In Section 2, we introduce the Schubert's normal forms of 1-bridge torus knots and classify some subfamily of 1-bridge torus knots expressed the Schubert's normal forms.

On the other hand, a 2-bridge knot can also be viewed as a 4-plats as studied first in [2]. From this side view, it is easy to see that the composition of homeomorphisms of a four-punctured sphere that determines the 2-bridge knot. Using this description, Conway constructed a bijection between 2-bridge knots and lens spaces via double branched covers [8]. A similar description using the composition of homeomorphisms on a two-punctured torus is possible for 1-bridge torus knots and this will be called the *Conway's normal form*. In Section 3, we construct the double branched cover of S^3 branched along an 1-bridge torus knot given by the Conway's normal form and give a formula for the first homology of the branched double cover.

2. SCHUBERT'S NORMAL FORMS

In this section, we introduce a notation describing a 1-bridge torus knot which is called Schubert's normal form and give a classification of subfamily of 1-bridge torus knots. The Schubert's normal form of a 1-bridge torus knot is an analogue of the Schubert's normal form of 2-bridge knot or link.

2.1. Schubert's normal forms.

Theorem 2.1. [6] *Any 1-bridge torus knots is represented by a 4-tuple $(r, s, t, \rho)_\epsilon$, where r, s, t are non-negative integers, ρ is an integer and ϵ is a sign ± 1 .*

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In the Schubert's normal form of a 1-bridge torus knot, r, s, t and ϵ determine the shape of the knot in the neighborhood of a meridian disk containing the bridge (See Figure 2), and ρ means the rotation number (See Figure 3).

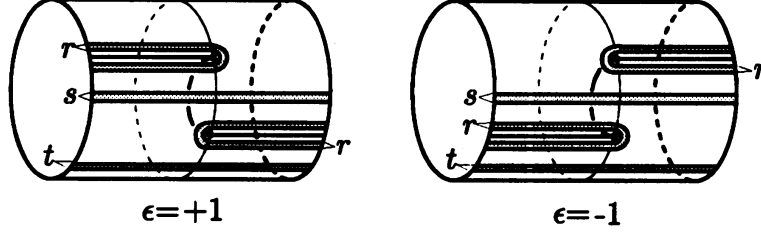


FIGURE 2

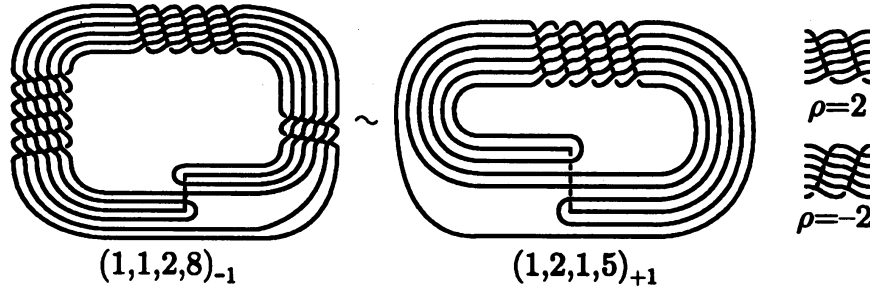


FIGURE 3. Schubert's normal forms of 1-bridge torus knots

Remark 2.2.

- (1) $(r, s, t, \rho)_{+1} = (r, t, s, \rho + (2r + 1))_{-1}$ (See Figure 3).
- (2) A 1-bridge torus knot with $(r, s, t, \rho)_{+1}$ is a mirror image of a 1-bridge torus knot with $(r, s, t, -\rho)_{-1}$.
- (3) If $r = 0$ in the normal form, then it represents a 1-bridge braid(See [10]).
- (4) A (p, q) -torus knot is a 1-bridge torus knot $(0, 0, p-1, -q)_{+1}$ or $(0, p-1, 0, -q+1)_{-1}$.
- (5) Any 2-bridge knot in S^3 has a Schubert's normal form $B(\alpha, \epsilon\beta)$ (See Chapter 3 of [1]), where

$$\alpha > 0, 0 < \beta < \alpha, \epsilon = \pm 1, \gcd(\alpha, \beta) = 1, \text{ and } \alpha, \beta \text{ odd.}$$

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A 2-bridge knot $B(\alpha, \epsilon\beta)$ is a 1-bridge torus knot $(\beta - 1, \alpha - 2\beta + 1, 0, \epsilon)_\epsilon$ (See Figure 4).

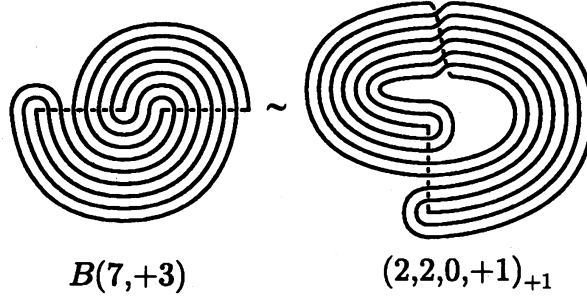
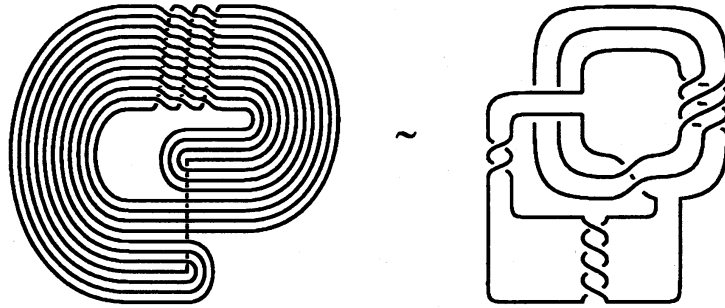


FIGURE 4

- (6) K. Morimoto and M. Sakuma showed that any satellite knot which admits an unknotting tunnel is equivalent to a knot represented by $K(\alpha, \epsilon\beta; p, q)$ in [13], where

α even integer, p, q positive integers, $\epsilon = \pm 1$ and $0 < \beta < \alpha/2$.

The knot $K(\alpha, \epsilon\beta; p, q)$ is a 1-bridge torus knot $(\frac{\beta - 1}{2}, \frac{\alpha - 2\beta}{2}, \frac{\alpha}{2}p, \frac{\alpha}{2}q)_\epsilon$.

FIGURE 5. $K = (3, 4, 0, -3)_{-1}$

2.2. sub-class $(r, s, 0, \epsilon(s - 1))_\epsilon$ of 1-bridge torus knots. Consider $(r, s, 0, \epsilon(s - 1))_\epsilon$, where $r \geq 0$, $s > 0$ are integers and $\epsilon = \pm 1$ (See Figure 5).

Lemma 2.3. $(r, s, 0, \epsilon(s - 1))_\epsilon$ is always the Schubert's normal form of 1-bridge torus knot. Furthermore, $(r, s, 0, -(s - 1))_{-1}$ is a mirror image of $(r, s, 0, s - 1)_{+1}$.

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Proof. For $(s-1, 0, 2(r+1), s)$, we get “1” from Component Counting Algorithm in [6], since $\gcd(1, 2r+s+1) = 1$. Therefore, $(r, s, 0, \epsilon(s-1))_\epsilon$ satisfies the conditions of Schubert’s normal form. \square

Since if $s = 1$ then $(r, 1, 0, 0)_\epsilon$ represents the unknot, we may assume that $s > 1$.

Theorem 2.4. [7] *Let $K_{r,s}$ be a 1-bridge torus knot $(r, s, 0, (s-1))_{+1}$. Then a genus of $K_{r,s}$ is*

$$\begin{cases} 2 + \frac{s(s-3)}{2} & \text{if } r \text{ is odd,} \\ \frac{s(s-1)}{2} & \text{if } r \text{ is even.} \end{cases}$$

Furthermore, $K_{r,s}$ is fibred if and only if $r = 0$ or 1.

Using Theorem 2.4, we get the following corollary;

Corollary 2.5. *$K_{r,s}$ is not isotopic to $K_{\bar{r},\bar{s}}$ if $r \neq \bar{r}$ or $s \neq \bar{s}$.*

Proof. Suppose $K_{r,s}$ is isotopic to $K_{\bar{r},\bar{s}}$.

Case 1) $r = \bar{r}$

If r and \bar{r} are odd then by Theorem 2.4,

$$2 + \frac{s(s-3)}{2} = g(K_{r,s}) = g(K_{\bar{r},\bar{s}}) = 2 + \frac{\bar{s}(\bar{s}-3)}{2}$$

Therefore, $s = \bar{s}$ or $s + \bar{s} = 3$. Since $s \neq \bar{s}$, $s + \bar{s} = 3$ and so s or \bar{s} is 1 but this is impossible, since $s, \bar{s} > 1$.

If r, \bar{r} are even then similarly, we meet a contradiction.

Case 2) $r \neq \bar{r}$

If r and \bar{r} are even(or odd), then by the method of Case 1, we meet a contradiction. So we may assume that r is odd and \bar{r} is even.

$$2 + \frac{s(s-3)}{2} = \frac{\bar{s}(\bar{s}-1)}{2}$$

Then integer solutions of the above equation are

$$\begin{pmatrix} s=1 \\ \bar{s}=2 \end{pmatrix}, \begin{pmatrix} s=1 \\ \bar{s}=0 \end{pmatrix}, \begin{pmatrix} s=1 \\ \bar{s}=-1 \end{pmatrix} \text{ and } \begin{pmatrix} s=2 \\ \bar{s}=2 \end{pmatrix}$$

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Therefore, the only possibility is the last solution. That is, $s = \bar{s} = 2$. Then $K_{r,s}$ (or $K_{\bar{r},\bar{s}}$) is a 2-bridge knot $B(2r + s + 1, r + 1)$ (resp. $B(2\bar{r} + \bar{s} + 1, \bar{r} + 1)$) (See 5. of Remark 2.2). Hence, $r = \bar{r}$. But this is impossible. \square

Theorem 2.6. [7]

$$\Delta_{r,s}(t) \doteq \begin{cases} \frac{(t-1)(t^{s(s+1)}-1)}{(t^s-1)(t^{s+1}-1)} + \frac{r(t-1)^2(t^{s^2-1}-1)}{2(t^{s+1}-1)} & \text{if } r \text{ is even,} \\ t^{(s^2-3s+4)/2} - \frac{r+1}{2} \frac{(t-1)^2(t^{(s-1)^2}-1)}{(t^{s-1}-1)} & \text{if } r \text{ is odd, } s = 2 \text{ or } 3, \\ \frac{t^{s-1}(t-1)(t^{(s-2)(s-1)}-1)}{(t^{s-2}-1)(t^{s-1}-1)} - \frac{r+1}{2} \frac{(t-1)^2(t^{(s-1)^2}-1)}{(t^{s-1}-1)} & \text{if } r \text{ is odd, } s \geq 4, \end{cases}$$

where $\Delta_{r,s}(t)$ is the Alexander polynomial of $K_{r,s}$.

Corollary 2.7. $K_{r,s}$ is fibred if and only if its Alexander polynomial is monic.

Proof. From Theorem 2.6,

$$\text{the leading coefficient of } \Delta_{r,s}(t) = \begin{cases} 1 + \frac{r}{2} & \text{if } r \text{ is even,} \\ -\frac{r+1}{2} & \text{if } r \text{ is odd,} \end{cases}$$

and $K_{r,s}$ is fibred iff $r = 0$ or 1 by Theorem 2.4. Hence, the proof is complete. \square

Recently, K. Murasugi and M. Hirasawa conjectured the above statement for twisted torus knots. They proved that it is true for the type 1:1 non-separable double torus knots and M. Hirasawa showed that the statement is also true for the sub-class of twisted torus knots. Therefore, their conjecture is true for our class.

Theorem 2.8. [7] For $r \geq 0, s \geq 2$,

$$(1) \quad V_{K_{r,s}}(t) = \frac{t^{3(s-1)\delta_r-s(s+1)/2}}{1-t^2} \left(\sum_{i=0}^{r-1} (-t^{-1})^i A(t) + (-t^{-1})^r B(t) \right) \text{ if } r \geq 1,$$

$$(2) \quad V_{K_{0,s}}(t) = \frac{t^{-s(s+1)/2}}{1-t^2} B(t),$$

where $A(t) = 1 - t^{s+2} - t^{2(2-s)} + t^{2-s}$ and $B(t) = 1 - t^{1-s} - t^{s+2} + t$.

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Corollary 2.9. [7] $K_{r,s}$ $r \geq 0$, $s \geq 2$ is non-amphicheiral except for $K_{1,2}$ which is a figure-eight knot.

Corollary 2.5 and Corollary 2.9 give us the classification of the 1-bridge torus knots with normal forms $(r, s, 0, \epsilon(s-1))_\epsilon$ ($r \geq 0$, $s \geq 2$).

Theorem 2.10. For any two 1-bridge torus knots K, K' with normal forms $(r, s, 0, \epsilon(s-1))_\epsilon$, $(r', s', 0, \epsilon'(s'-1))_{\epsilon'}$, respectively,

K is not isotopic to K' if $\epsilon \neq \epsilon'$, $r \neq r'$ or $s \neq s'$

except for $(1, 2, 0, 1)_{+1} = (1, 2, 0, -1)_{-1}$ which is a figure-eight knot.

3. CONWAY'S NORMAL FORMS AND DOUBLE BRANCHED COVERS

In this section, we concern about double branched covers of a 3-sphere branched along the 1-bridge torus knots. In order to this, we use an analogue of Conway's normal form of the 2-bridge knot (See Chapter 12. in [1] and Chapter 10. in [16]).

3.1. Conway's normal forms of 1-bridge torus knots. Let K be a 1-bridge torus knot, $(V_1, t_1) \cup_h (V_2, t_2)$ a $(1,1)$ -decomposition of (S^3, K) and \bar{t}_2 be an arc on ∂V_2 such that $t_2 \cup \bar{t}_2$ bounds a disk in V_2 . Then h is a homeomorphism from ∂V_2 onto ∂V_1 which is isotopic to a homeomorphism h_0 sending a meridian (resp. longitude) in ∂V_2 to a longitude (resp. meridian) in ∂V_1 , and $t_1 \cup h(\bar{t}_2)$ is isotopic to K . Then $\bar{h} = hh_0^{-1}$ is a homeomorphism on a torus isotopic to the identity.

The mapping class group $M(1,2)$ of a two-punctured torus is generated by d_m , d_ℓ , τ_ℓ , τ_m and σ (For example, see Chapter 4. in [5]), where d_m (resp. d_ℓ) is a Dehn-twist along the meridian (resp. longitude), σ is a homeomorphism exchanging two punctures and τ_m (resp. τ_ℓ) is a homeomorphism sliding one of punctures along the meridian (resp. longitude) as illustrated in Figure 6. By forgetting the punctures, a homomorphism $j_*: M(1,2) \rightarrow M(1,0)$ into the mapping class group of a torus is induced. Then \bar{h} is in $\ker j_*$. Therefore we can say that an element of $\ker j_*$ represents a $(1,1)$ -decomposition of a 1-bridge torus knot.

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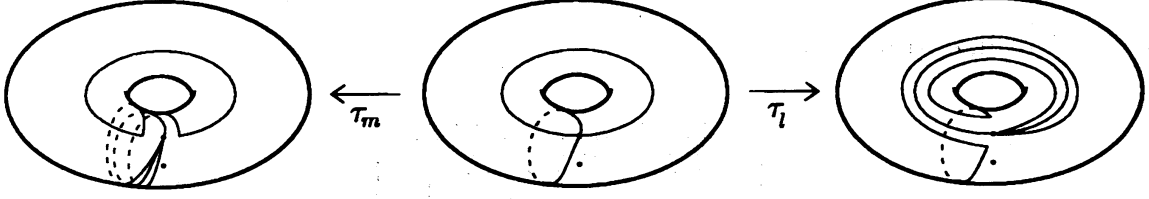


FIGURE 6

Consider the homeomorphisms $h_\ell = \tau_\ell \sigma \tau_\ell^{-1}$ and $h_m = \tau_m \sigma^{-1} \tau_m^{-1}$. Then

$$\tau_\ell^2 = h_\ell \sigma \quad \text{and} \quad \tau_m^2 = h_m^{-1} \sigma.$$

The homeomorphisms h_ℓ and h_m have an effect on the arc t_1 in V_1 as illustrated in Figure 7.

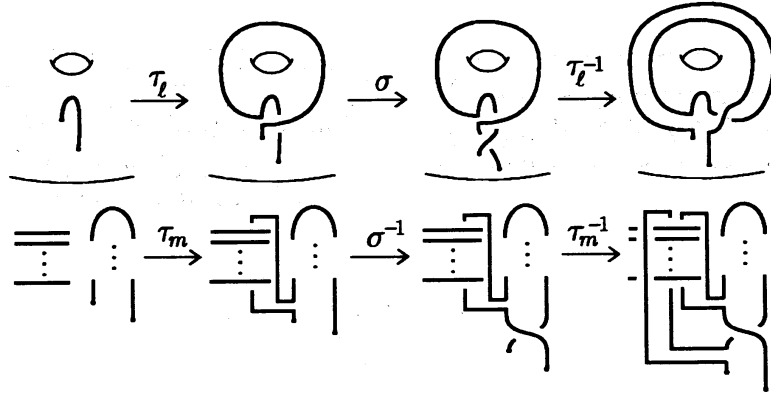


FIGURE 7

For integers $a_1, \dots, a_m, b_1, \dots, b_m$,

$$(3) \quad [(a_1, b_1, a_2, b_2), (a_3, b_3, a_4, b_4), \dots, (a_{m-1}, b_{m-1}, a_m, b_m)]$$

represents a 1-bridge torus knot in S^3 that has a (1,1)-decomposition $(V_1, t_1) \cup_h (V_2, t_2)$ such that $h = \bar{h} h_0$ and

$$\bar{h} = (h_\ell^{a_1} \sigma^{b_1} h_m^{a_2} \sigma^{b_2}) (h_\ell^{a_3} \sigma^{b_3} h_m^{a_4} \sigma^{b_4}) \dots (h_\ell^{a_{m-1}} \sigma^{b_{m-1}} h_m^{a_m} \sigma^{b_m}).$$

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The above $(1,1)$ -decomposition of a 1-bridge torus knot will be called a *Conway's normal form* of a 1-bridge torus knot.

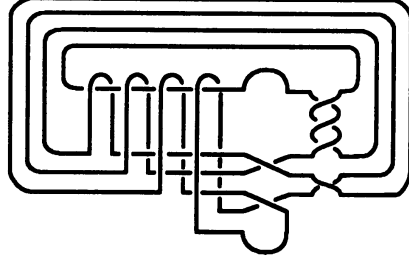


FIGURE 8. Conway's normal form $[(3, 0, 1, 0), (-1, 0, 1, 0)]$

Theorem 3.1. [6] *Every 1-bridge torus knot has a Conway's normal form.*

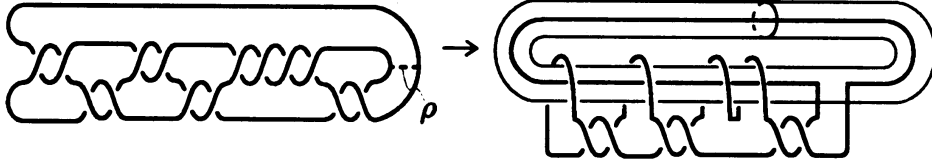


FIGURE 9

Remark 3.2. A 2-bridge knot has the Conway's normal form $[2a_1, 2a_2, \dots, 2a_m]$ as illustrated in Figure 9. We choose a $(1,1)$ -tunnel ρ as in Figure 9. Then we get a $(1,1)$ -decomposition of it and the attaching homeomorphism of the $(1,1)$ -decomposition is $h_0(\tau_\ell \sigma^{-1})(\sigma^{-2a_m} \tau_m^{a_m-1}) \dots (\sigma^{-2a_2} \tau_m^{a_1})$ (See Figure 9). By using the relations of $\ker j_*$ we can obtain a Conway's normal form of 1-bridge torus knot for the given 2-bridge knot.

3.2. Double branched covers along 1-bridge torus knots. Consider a double branched cover Σ of a solid torus V branched along a trivial arc in V , which is a genus two handlebody (See Figure 10).

Then from Figure 10 and Figure 11, the following facts are evident;

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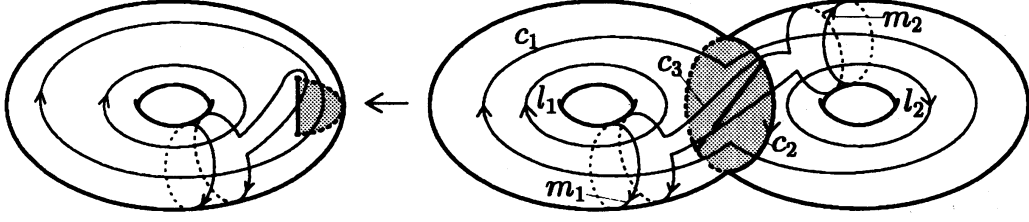
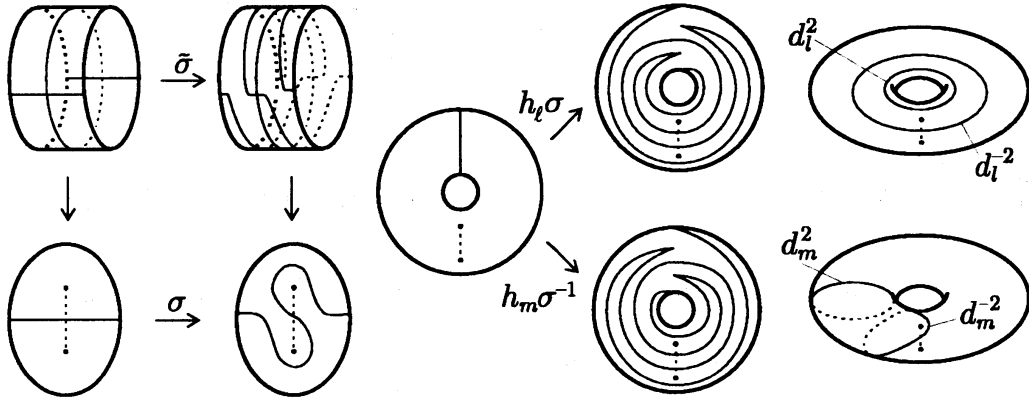


FIGURE 10. Double branched cover of a solid torus branched along an arc

- (1) The lifting of h_0, \tilde{h}_0 , is a homeomorphism of $\partial\Sigma$ such that $\tilde{h}_0(m_1) = l_1, \tilde{h}_0(m_2) = l_2, \tilde{h}_0(l_1) = m_1$ and $\tilde{h}_0(l_2) = m_2$.
- (2) The lifting of $\sigma, \tilde{\sigma}$, is d_{c_2} , where c_2 is a curve as shown in Figure 10.
- (3) The lifting of h_ℓ, \tilde{h}_ℓ , is $d_{c_1}^{-1} d_{l_1}^2 d_{l_2}^2 d_{c_2}^{-1}$,
- (4) The lifting of h_m, \tilde{h}_m , is $d_{c_3}^{-1} d_{m_1}^2 d_{m_2}^2 d_{c_2}$, where $c_i (i = 1, 2, 3)$ is a curve depicted at Figure 10.

FIGURE 11. The lifting of σ and the homeomorphisms h_ℓ, h_m

Therefore, we can obtain the following theorem;

Theorem 3.3. *If a 1-bridge torus knot K has a Conway's normal form*

$$[(a_1, b_1, a_2, b_2), (a_3, b_3, a_4, b_4), \dots, (a_{m-1}, b_{m-1}, a_m, b_m)]$$

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then the double branched cover X_2 of S^3 branched along K has a genus two H splitting $\Sigma_1 \cup_{\tilde{h}} \Sigma_2$ such that $\Sigma_i (i = 1, 2)$ is a genus two handlebody and

$$\tilde{h} = (\tilde{h}_\ell^{a_1} \tilde{\sigma}^{b_1} \tilde{h}_m^{a_2} \tilde{\sigma}^{b_2}) (\tilde{h}_\ell^{a_3} \tilde{\sigma}^{b_3} \tilde{h}_m^{a_4} \tilde{\sigma}^{b_4}) \cdots (\tilde{h}_\ell^{a_{m-1}} \tilde{\sigma}^{b_{m-1}} \tilde{h}_m^{a_m} \tilde{\sigma}^{b_m}) \tilde{h}_0.$$

Lemma 3.4. [6]

$$\begin{aligned} \tilde{h}_*([m_1]) &= 1/2 z_m [m_1] + (a_m z_m + 1/2(z_{m-1} + 1)) [l_1] \\ &\quad - 1/2 z_m [m_2] - (a_m z_m + 1/2(z_{m-1} - 1)) [l_2], \end{aligned}$$

where z_m is a sequence such that $z_m = 2a_{m-1}z_{m-1} + z_{m-2}$, $z_0 = 0$ and $z_1 = 1$.

Proposition 3.5. Let z_m be a sequence satisfying the following recursive formula;

$$z_{m+1} = 2a_m z_m + z_{m-1}, z_0 = 0 \text{ and } z_1 = 1,$$

where a_m is a sequence. Then

$$(4) \quad z_{m+1} = 2^m (a_1 a_2 \cdots a_m) + \sum_{t=1}^{\lfloor \frac{m}{2} \rfloor} 2^{m-2t} \sum_{\substack{(j_1, \dots, j_t) \\ \in C_m^t}} A(j_1, j_2, \dots, j_t),$$

where

$$C_m^t = \{(j_1, \dots, j_t) \in \mathbf{N}^t \mid 1 \leq j_1 < \cdots < j_t < m, j_k - j_{k-1} \geq 2, k = 1, \dots, t\}$$

$$A(j_1, j_2, \dots, j_t) = (a_1 a_2 \cdots a_{j_1-1}) (a_{j_1+2} \cdots a_{j_2-1}) \cdots (a_{j_t+2} \cdots a_m),$$

and $A(1, 3, \dots, m-1) = 1$ when m is even.

From Lemma 3.4 and Proposition 3.5, we calculate the first homology of X_2 .

Theorem 3.6. Let K be a 1-bridge torus knot with the Conway's normal form

$$[(a_1, b_1, a_2, b_2), (a_3, b_3, a_4, b_4), \dots, (a_{m-1}, b_{m-1}, a_m, b_m)],$$

and X_2 be a double branched cover of S^3 branched along K . Then $H_1(X_2) = \mathbf{Z}/|z|$, where z_{m+1} is a sequence at the formula (4).

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Proof. By excision, $H_1(X_2) \cong H_1(\Sigma_1 \cup (B_1 \cup B_2))$, where B_1, B_2 are the tubular neighborhoods of meridian disks D_1, D_2 of Σ_2 . And by Mayer-Vietoris sequence, $H_1(X_2) = \text{Coker}(f: H_1(A_1 \cup A_2) \rightarrow H_1(\Sigma_1))$, where $A_i = \partial D_i \times I (i = 1, 2)$. Since $[m_1]$ and $[m_2]$ generate $H_1(A_1 \cup A_2)$ and $H_1(\Sigma_1) = \langle l_i, m_i \mid m_i = 0, i = 1, 2 \rangle$, $f([m_i]) = \tilde{h}_*([m_i]), i = 1, 2$. From Lemma 3.4 and the periodic property of X_2 ,

$$\begin{aligned} f([m_1]) &= (a_m z_m + 1/2(z_{m-1} + 1))[l_1] - (a_m z_m + 1/2(z_{m-1} - 1))[l_2], \\ f([m_2]) &= -(a_m z_m + 1/2(z_{m-1} - 1))[l_2] + (a_m z_m + 1/2(z_{m-1} + 1))[l_1]. \end{aligned}$$

Therefore, $H_1(X_2) = \langle l_1, l_2 \mid R \rangle$, where

$$R = \begin{bmatrix} (a_m z_m + 1/2(z_{m-1} + 1)) & -(a_m z_m + 1/2(z_{m-1} - 1)) \\ -(a_m z_m + 1/2(z_{m-1} - 1)) & (a_m z_m + 1/2(z_{m-1} + 1)) \end{bmatrix}.$$

Hence, the proof is complete since

$$R \sim \begin{bmatrix} 1 & 0 \\ 0 & 2a_m z_m + z_{m-1} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & z_{m+1} \end{bmatrix}$$

□

Corollary 3.7. $H_1(X_2)$ is a finite cyclic group and $|z_{m+1}| = |\Delta_K(-1)|$, where $\Delta_K(t)$ is the Alexander polynomial of K .

Corollary 3.8. Suppose K is a 1-bridge torus knots with the Conway's normal form

$$[(a_1, b_1, a_2, b_2), (a_3, b_3, a_4, b_4), \dots, (a_{m-1}, b_{m-1}, a_m, b_m)].$$

- (1) If $a_{2i} = 0$ or $a_{2i-1} = 0$ for $i = 1, \dots, m/2$ then K is a trivial knot.
- (2) If either $a_i > 0$ or $a_i < 0$ then K is not a trivial knot.

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DEPARTMENT OF MATHEMATICS, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY,
TAEJON 305–701, KOREA

E-mail address: dhchoi@knot.kaist.ac.kr

DEPARTMENT OF MATHEMATICS, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY,
TAEJON 305–701, KOREA

E-mail address: knot@knot.kaist.ac.kr